Solutions

1.4: Logic in Mathematics

In this section we provide an overview of the basic mathematical "parts of speech."

Question 1. Explain why an integer cannot be both even and odd.

2 divides all even numbers, whereas 2 does not divide any odd number. Since 2 cannot both divide and not divide any integer, an integer cannot be both even and odd.

Question 2. Recall that a *prime number* is a natural number n such that $n > 1$ and n has no divisors other than n and 1. Prove or disprove the following: Every prime number greater than 3 is the sum of two prime numbers.

This statement is false.

17 is a prime number that cannot be written as the sum of two prime numbers.

<u>**Definitions in Mathematics:**</u> In language, the purpose of a definition is to *describe* something that already exists. In mathematics, a **definition** is a statement that stipulates the meaning of a *new* term, symbol, or object. For example,

> **Definition 1.** The lines are **parallel** if they have no points in common.

The job of this definition is to specify exactly what is meant when we use the word "parallel." It does not describe parallel lines. If you were to look up the word "parallel" in the dictionary, you may find the statement "side by side and having the same distance continuously between them." Whereas this intuitive description may be true, and it is in certain geometries, the only thing we know about parallel lines is that they don't intersect (unless we prove something else about them).

Definition 2. An integer n is **even** if $n = 2k$ for some integer k.

Definition 3. An integer n is **odd** if $n = 2k + 1$ for some integer k.

Question 3. Given these definitions can we revisit Question 1 and *prove* that an integer cannot be both even and odd?

Kind of. It is "clear" that $2K_0 = 2K_1 + 1$ should not be solveable. However, we arrive at $1 = 2(K_0 - K_1)$ so one is even. But $1 = 2 \cdot 0 + 1$, so 1 is odd. So 1 is both even and odd. This becomes circular logic.

Since definitions are stipulative, the are logically "if and only if" statements. However, it is common to write definitions of the form

$$[\text{Object}]\ x \text{ is } [\text{defined term}] \text{ if } [\text{defining property about } x].$$

All proceeding definitions are of this form. In predicate logic, if

$$\begin{aligned} D(x) &= x \text{ is } [\text{defined term}] \\ P(x) &= [\text{defining property about } x] \end{aligned}$$

then the above definition really means

$$(\forall x)(P(x) \leftrightarrow D(x)).$$

Other Mathematical Statements: This method of defining everything specifically has to stop somewhere. Consider the geometric definitions given by Euclid in "The Elements:"

1. A point is that which has no breadth. 2. A line is a breadthless length.

What is meant by "breadth" or "length?" Intuitively we know, but not mathematically. In modern mathematics we recognize this fallacy and call those statements which are assumed without proof either **postulates** or **axioms.** So any statement that we prove to be "true" is not true in the absolute sense, only given some specified set of axioms.

A **theorem** is a statement that follows logically from axioms and other statements we have already proven. A **proof** is a valid argument if and only if it follows only from the axioms, definitions and already established theorems. A **lemma, proposition**, or **corollary** is a specific type of theorem.

Counterexamples: Consider Question 2. We were asked to prove or disprove a statement of the form

$$(\forall x)P(x).$$

Proving this statement could be difficult, as we have to show something for every valid input. However, to disprove the statement, we need only prove that the negation of the statement holds; i.e.

$$(\exists x)\neg P(x).$$

Therefore we need only find a single example of when the statement fails. This is called a **counterexample.** An **example** is used to prove a "there exists" statement. The adjective counter is used to emphasize we are giving an example of a negation.

Example 1. Disprove the statement "every prime number is odd."

"$(\forall x)P(x)$"

$\cancel{\neg P(x)}$: $\neg(\forall x)P(x) \Leftrightarrow (\exists x)\neg P(x) \Leftrightarrow$ there exists some prime number which is even.

2 is an example of such a prime number.

Another common form in mathematics is the universal if-then statement.

$$(\forall x)(P(x) \rightarrow Q(x)).$$

The negation of this statement is

$$(\exists x)(P(x) \wedge \neg Q(x)).$$

Example 2. Disprove the following statement.

For all sequences of numbers $a_1, a_2, a_3, \ldots$, if $a_1 < a_2 < a_3 < \cdots$, then some a_i must be positive.

$\neg(\forall x)(P(x) \rightarrow Q(x)) \Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$.

The sequence $a_i = -\frac{1}{i}$ is such a sequence.

Axiomatic System for a Four-Point Geometry:

Undefined terms: point, line, is on
Axioms:

1. For every pair of distinct points x and y, there is a unique line l such that x is on l and y is on l.
2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m and no point on l is also on m.
3. There are exactly four points.
4. It is impossible for three points to be on the same line.

Definition 4. A line l **passes through** x if x is on l. **Definition 5.** Two line, l and m, are **parallel** if there is no point x, such that x is on l and x is on m.

Theorem 1. *There are at least two distinct lines.*

Proof. By Axiom 3, there are distinct points x, y, and z.
By Axiom 1, there is a line l through x and y,
and a line m through y and z.
By Axiom 4, x, y and z are not on the same
line, so l and m must be distinct.

Homework. (Due Oct 1, 2018) Section 1.4: 6, 10, 13

Practice Problems. Section 1.4: 5, 8, 9, 19-24